Traffic Modeling and Real-time Control for Metro Lines

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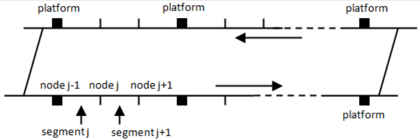
Outline

1. Max-plus algebra model
   - analytical results on the tain dynamics
   - without taking into account passenger traffic demand
2. Stochastic dynamic programming model
   - Take into account passenger demand
3. Fix the parameters of the control model
   - Derivation of the passenger demand effect on the train dynamics and on the service performance

Max-plus algebra model



n number of segments in the line.

m number of moving trains.

d_j^k the kth train departure time from node j.

a_j^k the kth train arrival time to node j.

r_j the running time of a train on segment j.

$w_j^k = d_j^k - a_j^k$ the kth train dwell time on node j.

$t_j^k = r_j + w_j^k$.

$g_j^k = a_j^k - d_j^{k-1}$ the node safe separation

$h_j^k = d_j^k - d_j^{k-1} = g_j^k + w_j^k$: the kth departure time-headway at node j.

$s_j^k = g_j^{k+b_j} - r_j^k$ (by definition).

$\bar{x}$ upperbound for x.

$\underline{x}$ lower bound for x.

$g = r + s$

$t = r + w$

$h = g + w = t + s = (n/m)t = (n/(n-m))s$

The model

Two time constraints

- A constraint on the travel time on every segment j.

$$d_j^k \geq d_{j-1}^{k-b_j} + \underline{t}_j. \tag{1}$$

- A constraint on the safe separation time at every segment j.

$$d_j^k - d_{j+1}^{k-\bar{b}_{j+1}} = a_{j+1}^{k+b_{j+1}} - r_{j+1} - d_{j+1}^{k-\bar{b}_{j+1}} = g_{j+1}^{k+b_{j+1}} - r_{j+1} \geq \underline{g}_{j+1} - r_{j+1} = \underline{s}_{j+1}.$$

That is

$$d_j^k \geq d_{j+1}^{k-\bar{b}_{j+1}} + \underline{s}_{j+1}. \tag{2}$$

- We combine the two constraints

$$d_j^k = \max\{d_{j-1}^{k-b_j} + \underline{t}_j, d_{j+1}^{k-\bar{b}_{j+1}} + \underline{s}_{j+1}\}, \; k \geq 1, 1 \leq j \leq n, \tag{3}$$

Max-plus algebra formulation

We have

$$d_j^k = \max\{d_{j-1}^{k-b_j} + \underline{t}_j, d_{j+1}^{k-\bar{b}_{j+1}} + \underline{s}_{j+1}\}, \; k \geq 1, 1 \leq j \leq n,$$

Max-plus operations and noations

- $a \oplus b := \max(a, b)$
- $a \otimes b := a + b$
- $\gamma x(k) := x(k-1)$
- $\gamma^l x(k) := x(k-l)$

Then

$$d_j = \underline{t}_j \gamma^{b_j} d_{j-1} \oplus \underline{s}_{j+1} \gamma^{\bar{b}_{j+1}} d_{j+1}, \quad 1 \leq j \leq n. \tag{4}$$

Max-plus matrix formulation

Homogeneous linear Max-plus algebra systems

$$x(k) = \bigoplus_{l=0}^{p} A_l \otimes x(k-l) = \bigoplus_{l=0}^{p} \gamma^l A_l x = A(\gamma)x. \quad (5)$$

Then

$$d = A(\gamma) \otimes d, \quad (6)$$

where $A(\gamma)$ is given as follows.

$$A(\gamma) = \begin{pmatrix} \varepsilon & \gamma^{\bar{b}_2}\underline{s}_2 & \varepsilon & \cdots & \varepsilon & \gamma^{b_1}\underline{t}_1 \\ \gamma^{b_2}\underline{t}_2 & \varepsilon & \gamma^{\bar{b}_3}\underline{s}_3 & \varepsilon & \cdots & \varepsilon \\ & \ddots & \varepsilon & \ddots & & \\ \varepsilon & \cdots & \gamma^{b_j}\underline{t}_j & \varepsilon & \gamma^{\bar{b}_{j+1}}\underline{s}_{j+1} & \varepsilon \\ & & & \ddots & \varepsilon & \\ \gamma^{\bar{b}_1}\underline{s}_1 & \varepsilon & \cdots & \varepsilon & \gamma^{b_n}\underline{t}_n & \varepsilon \end{pmatrix}$$

Max-plus generalized eigenvalue

Generalized eigenvalue

$$A(\mu^{-1}) \otimes v = v, \quad (7)$$

where $A(\mu^{-1})$ is the matrix obtained by valuating the polynomial matrix $A(\gamma)$ at μ^{-1}.

Theorem (Baccelli et al. 1992, Goverd 2007)
Let $A(\gamma) = \oplus_{l=0}^{p} A_l \gamma^l$ be an irreducible polynomial matrix with acyclic subgraph $\mathcal{G}(A_0)$. Then $A(\gamma)$ has a unique generalized eigenvalue $\mu > \varepsilon$ and finite eigenvectors $v > \varepsilon$ such that $A(\mu^{-1}) \otimes v = v$, and μ is equal to the maximum cycle mean of $\mathcal{G}(A(\gamma))$.

$$\mu = \max_{c \in \mathcal{C}} W(c)/D(c),$$

where $\mathcal{C}$ is the set of all elementary circuits in $\mathcal{G}(A(\gamma))$.

Graph associated to $A(\gamma)$

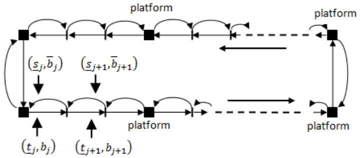
- For every $0 \leq l \leq p$, an arc (i, j, l) is associated for each non-null ($\neq \varepsilon$) entree (i, j) of max-plus matrix A_l.
- A *weight* $W(i, j, l)$ and a *duration* $D(i, j, l)$ are associated to each arc (i, j, l) in the graph, with $W(i, j, l) = (A_l)_{ij} \neq \varepsilon$ and $D(i, j, l) = l$.
- Similarly, a weight, resp. duration of a circuit (directed cycle) in the graph is the standard sum of the weights, resp. durations of all the arcs of the circuit.
- The cycle mean of a circuit c with a weight $W(c)$ and a duration $D(c)$ is $W(c)/D(c)$.
- A polynomial matrix $A(\gamma)$ is said to be irreducible, if $\mathcal{G}(A(\gamma))$ is strongly connected.

Average asymptotic train time-headway

Theorem

The dynamic system admits a unique additive eigenvalue μ, which is also its asymptotic average growth rate, and which is interpreted in term of train dynamics, as the asymptotic average time-headway h of the trains. We have

$$h = \mu = \max \left\{ \frac{\sum_j \underline{t}_j}{m}, \max_j(\underline{t}_j + \underline{s}_j), \frac{\sum_j \underline{s}_j}{n-m} \right\}.$$

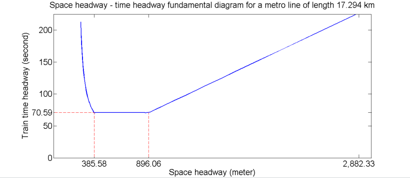


Proof

h is given as the maximum cycle mean of $\mathcal{G}(A(\gamma))$. Three different elementary circuits are distinguished on $\mathcal{G}(A(\gamma))$.

- The hamiltonian circuit in the direction of the train movements, with mean $\sum_j \underline{t}_j / m$.
- All the circuits of two links relying nodes j and $j+1$, with mean $\underline{t}_j + \underline{s}_j$ each.
- The hamiltonian circuit in reverse direction of the train dynamics, with mean $\sum_j \underline{s}_j / (n-m)$.

Fundamental traffic diagram (1/3)

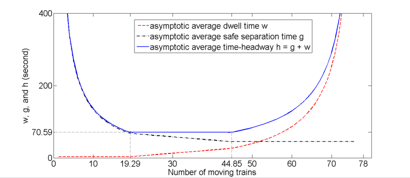


- 9 stations ⇒ 18 platforms
- max train speed = 80 km/h
- block length = 200 m
- minimum dwell time = 10 s
- safety time = 30 s

$$h(\sigma) = \max \left\{ \tau\sigma, h_{\min}, \frac{\omega}{\frac{1}{\underline{\sigma}} - \frac{1}{\sigma}} \right\},$$

- h is the average time headway,
- $\sigma = L/m$ is the average space-headway,
- $\tau = \sum_j \underline{t}_j / L = 1/v$ is the inverse of the maximum train speed v,
- $h_{\min} = \max_j h_j = \max_j (\underline{t}_j + \underline{s}_j)$,
- $\omega = \sum_j \underline{s}_j / L$,
- $\underline{\sigma} = L/n$ is the minimum space-headway of trains on the line.

Fundamental traffic diagram (2/3)



- 9 stations $\Rightarrow$ 18 platforms
- max train speed = 80 km/h
- block length = 200 m
- minimum dwell time = 10 s
- safety time = 30 s

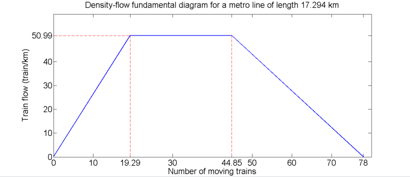
$$h(\sigma) = \max\left\{ \tau\sigma, h_{\min}, \frac{\omega}{\frac{1}{\underline{\sigma}} - \frac{1}{\sigma}} \right\},$$

$$w(\rho) = \max\left\{ \underline{w}, \frac{h_{\min}}{\bar{\rho}}\rho - r, \frac{\omega}{\bar{\rho} - \rho} - \underline{g} \right\}.$$

$$g(\rho) = \max\left\{ \frac{\tau}{\rho} - \underline{w}, (r + h_{\min}) - \frac{h_{\min}}{\bar{\rho}}\rho, \underline{g} \right\}.$$

- $\bar{\rho} = 1/\underline{\sigma}$ is the maximum train density on the line.
- $\underline{w} = \sum_j \underline{w}_j / n$, $r = \sum_j r_j / n$.
- $\underline{g} = \sum_j \underline{g}_j / n$.

Fundamental traffic diagram (3/3)

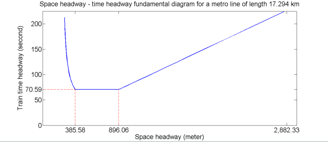


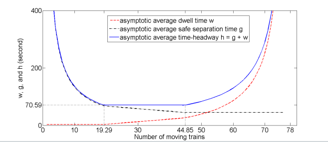
- 9 stations $\Rightarrow$ 18 platforms
- max train speed = 80 km/h
- block length = 200 m
- minimum dwell time = 10 s
- safety time = 30 s

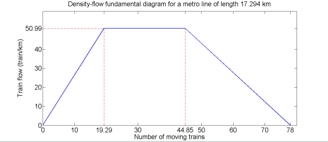
$$f(\rho) = \min\left\{v\rho, f_{\max}, w'(\bar{\rho} - \rho)\right\},$$

- $f_{\max} = 1/h_{\min}$ is the maximum flow of trains that can pass through one segment.
- $v = 1/\tau$ is the free (or maximum) train-speed on the metroline.
- $w' = 1/\omega$ is the backward wave-speed for the train dynamics.

The traffic phases (1/3)



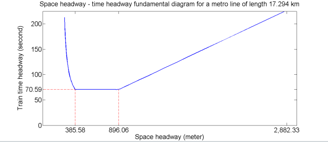


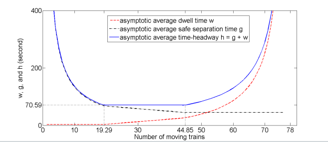


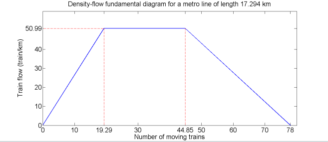
Free flow traffic phase. $(0 \leq \rho \leq f_{\max}/v)$

- Trains move freely on the line, which operates under capacity.
- Big average time-headways $h = w + g$.
- Minimum dwell times w.
- Big average safe separation time g.

The traffic phases (2/3)





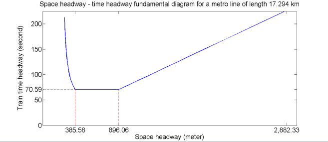


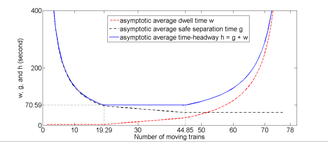
Maximum train-capacity traffic phase.

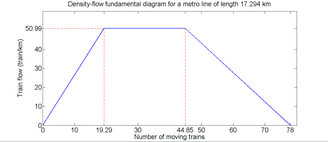
$(f_{\max}/v \leq \rho \leq \bar{\rho} - f_{\max}/w')$.

- The line operates at its maximum train-capacity.
- The frequency is independent of the number of moving trains.
- The average dwell time w increases linearly with the number of the moving trains.
- The average safe separation time g decreases linearly with the number of moving trains.
- The average time-headway $h = g + w$ remains constant and independent of the number of moving trains.
- The optimum number $Lf_{\max}/v$ of moving trains on the line is attained at the beginning of the this traffic phase, as the passengers are not taken into account.

The traffic phases (3/3)



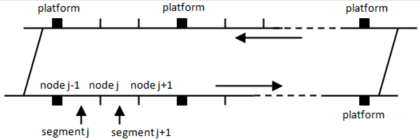




Congestion traffic phase. $(\bar{\rho} - f_{\max}/w' \leq \rho \leq \bar{\rho})$.

- Trains bother each other on the line, which operates under capacity.
- Big average time-headways *h*.
- The safe separation time is independent of the number of moving trains.
- The average time-headway, as well as the average dwell time increase rapidly with the number of moving trains.

Unstable dynamic model (1/3)



- The train dwell time w_j at platform j depends on the passenger volume at platform j.
- which depends on the safe separation time g_j on the same platform.
- We do not consider a dynamic model for the passenger volumes on platforms.
- The dwell times on platforms depend directly on the passenger arrival rates.

We consider the following additional constraint on the dwell time at plaforms.

$$w_j^k \geq \begin{cases} \frac{\lambda_j}{\alpha_j} g_j^k, & \text{if } j \text{ indexes a platform,} \\ 0 & \text{otherwise.} \end{cases}$$

- α_j is the total passenger upload rate from platform j onto trains, if j indexes a platform ; and α_j is zero otherwise.
- $g_j^k = a_j^k - d_j^{k-1}$ is, the safe separation time on segment j.
- λ_j is the average rate of the total arrival flow of passengers to platform j, if j indexes a platform ; and λ_j is zero otherwise.

Unstable dynamic model (2/3)

By taking ito account the additional constraint we have

$$d_j^k \geq d_{j-1}^{k-b_j} + r_j + \max\left\{ \underline{w}_j, \frac{\lambda_j}{\alpha_j} g_j^k \right\}.$$

Then

$$d_j^k = \max \begin{cases} d_{j-1}^{k-b_j} + r_j + \underline{w}_j, \\ \left(1 + \frac{\lambda_j}{\alpha_j}\right) d_{j-1}^{k-b_j} - \left(\frac{\lambda_j}{\alpha_j}\right) d_j^{k-1} + \left(1 + \frac{\lambda_j}{\alpha_j}\right) r_j, \\ d_{j+1}^{k-\bar{b}_{j+1}} + \underline{s}_{j+1}. \end{cases}$$

- The dynamic system has explicit and implicit terms.
- It can be written as follow.

$$d_j^k = \max_{u \in \mathcal{U}} [(M^u d^{k-1})_j + (N^u d^k)_j + c_j^u],$$

  where M^u and N^u are square matrices, and c^u is a family of vectors.
- Matrices N^u express implicit terms.
- If $\exists j, \lambda_j / \alpha_j > 0$, then one of the matrices $M^u, u \in \mathcal{U}$ or $N^u, u \in \mathcal{U}$ is not sub-stochastic.
- In this case, the dynamic system cannot be seen as a dynamic programming system of a stochastic optimal control problem.

Unstable dynamic model (3/3)

Particular cases

- If $m = 0$ or $m = n$, then the dynamic system is fully implicit (it is not triangular).
- It admits an asymptotic regime with a unique asympotic average train time-headway.
- This case corresponds to 0 or n trains on the metro line. No train departure is possible for these two cases.
- We have the average train flow $f = 0$ corresponding to the average time headway $h = +\infty$.

- If $0 < m < n$, then the dynamic system is triangular.
- There exists an order of updating the components of the state vector d^k, in such a way that no implicit term appears.

Instability

- The dynamic system is not stable (see Breusegem et al. 1991).
- Consider the metro line as a server of passengers.
  - average passenger arrival rate λ
  - average service rate $\alpha w^* / h$ (with the assumption of infinite passenger capacity of trains).
- In the high passenger demand case, where the second term of the maximum operator of the dynamics is activated, we get

$$w^* = (\lambda/\alpha)g$$

- Therefore $\lambda = \alpha w^* / g > \alpha w^* / h$ since $g < h$.
- Hence, the passenger server is unstable.

Stable dynamic programming model (1/3)

- We modify the train dynamics in order to guarantee its stability.
- We replace the dwell time control formula

$$w_j^k \geq \begin{cases} \frac{\lambda_j}{\alpha_j} g_j^k, & \text{if } j \text{ indexes a platform}, \\ 0 & \text{otherwise}. \end{cases}$$

by the following.

$$w_j^k \geq \begin{cases} \overline{w}_j - \frac{\theta_j^k}{\lambda_j^k / \alpha_j^k} g_j^k & \text{if } j \text{ indexes a platform}, \\ 0 & \text{otherwise}. \end{cases}$$

- We reversed the sign of the relationship between the dwell time w_j^k and the safe separation time g_j^k.
- without reversing the relationship between the dwell time w_j^k and the ratio λ_j^k / α_j^k.
- $\overline{w}_j$ (maximum dwell time on node j) and θ_j^k are control parameters to be fixed.

- The dynamics are now written

$$d_j^k = \max \begin{cases} d_{j-1}^{k-b_j} + r_j + \underline{w}_j, \\ \left(1 - \delta_j^k\right) d_{j-1}^{k-b_j} + \delta_j^k d_j^{k-1} + \left(1 - \delta_j^k\right) \\ d_{j+1}^{k-\bar{b}_{j+1}} + \underline{s}_{j+1}, \end{cases}$$

where $\delta_j^k = \theta_j^k \alpha_j^k / \lambda_j^k, \forall j, k$.

- If δ_j^k are independent of k for every j, then

$$d_j^k = \max_{u \in \mathcal{U}} [(M^u d^{k-1})_j + (N^u d^k)_j + c_j^u],$$

- In this case, the system is a dynamic programming system of an optimal control problem of a Markov chain.

Stable dynamic programming model (2/3)

Let us consider the dynamic system $x^{k+1} = f(x^k)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$.

- A map f is said to be additive 1-homogeneous if

$$\forall x \in \mathbb{R}^n, \forall a \in \mathbb{R}, f(a\mathbf{1} + x) = a\mathbf{1} + f(x).$$

- f is said to be monotone if

$$\forall x, y \in \mathbb{R}^n, x \leq y \Rightarrow f(x) \leq f(y).$$

- If f is 1-homogeneous and monotone, then it is non expansive (or 1-Lipschitz) for the supremum norm, i. e.

$$\forall x, y \in \mathbb{R}^n, ||f(x) - f(y)||_\infty \leq ||x - y||_\infty,$$

- In this case a directed graph $\mathcal{G}(f)$ is associated to f.

Directed graph $\mathcal{G}(f)$ associated to f (Gaubert et al. 1999)
It is defined by the set of nodes $\{1, 2, \ldots, n\}$ and by a set of arcs such that there exists an arc from a node i to a node j if $\lim_{\eta\to\infty} f_i(\eta e_j) = \infty$, where e_j is the jth vector of the canonical basis of $\mathbb{R}^n$.

Theorem. (Gaubert et al. 1998, Gunawardena et al. 1995)

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is 1-homogeneous and monotone and if $\mathcal{G}(f)$ is strongly connected then

- f admits an (additive) eigenvalue, i.e.

$$\exists \mu \in \mathbb{R}, \exists x \in \mathbb{R}^n : f(x) = \mu + x.$$

- Moreover, μ coincides with the asymptotic average growth rate of the dynamic system $x^{k+1} = f(x^k)$, defined by $\lim_{k\to\infty} f^k(x)/k$.

Stable dynamic programming model (3/3)

- The system of the stationary regime is

$$h + d_j = \max_{u \in \mathcal{U}} [(M^u d)_j + (N^u (h + d))_j + c_j^u],$$

where h is an eigenvalue and d is an associated eigenvector.

Theorem.

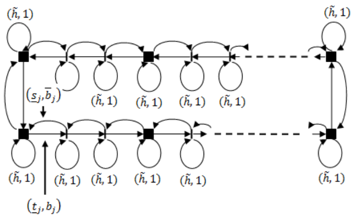
1. If δ_j^k are independent of k for every j, and if $0 \leq \delta_j \leq 1, \forall j$, then the algebraic system of the stationary regime admits a unique eigenvalue h.
2. Moreover, the asymptotic average train time-headway, coincides with the eigenvalue h, independent of the initial state vector d^0.

- We do not have yet an analytical formula for the asymptotic train time-headway.
- However, the Theorem above guarantees its existence and its uniqueness.
- Therefore, by iterating the dynamics, one can approximate the asymptotic average train time-headway as follows.

$$h \approx d_j^K / K, \forall j, \text{ for a big value of } K.$$

Control parameters $\bar{w}$ and θ

Theorem. Let $\tilde{h}$ be the asymptotic average time-headway solution of the max-plus linear system. The dynamic programming system with parameters $\bar{w}_j = \tilde{h}, \forall j$, and $\delta_j = 1, \forall j$, is a max-plus linear system, whose asymptotic average time-headway coincides with $\tilde{h}$.



$$h = \max \left\{ \frac{\sum_j \underline{t}_j}{m}, \max_j(\underline{t}_j + \underline{s}_j), \frac{\sum_j \underline{s}_j}{n-m}, \tilde{h} \right\} = \tilde{h}.$$

proof.

- If $\delta_j = 1, \forall j$, then system is a max-plus linear system whose associated graph has n additional circuits (which are loop-circuits).
- Moreover, if $\bar{w}_j = \tilde{h}, \forall j$, then the cycle mean of the loops are all equal to $\tilde{h}$.
- All the other parameters remaining the same, the asymptotic average time-headway h is given by the maximum cycle mean of the graph associated to the obtained max-plus linear system.
- Four different elementary circuits are distinguished.
  - We have the same three cicuits.
  - The n additional loop-circuits have mean $\tilde{h}$.

Control parameters $\bar{w}$ and θ

- The latter Theorem, tells us that one can simply fix

$$(\bar{w}_j, \theta_j^k) = (\tilde{h}(\rho), \lambda_j^k / \alpha_j^k)$$

or equivalently

$$(\bar{w}_j, \delta_j^k) = (\tilde{h}(\rho), 1),$$

to obtain a max-plus linear dynamic system, which does not take into account passenger effects.

- The question here, is rather, how to fix the control parameters $(\bar{w}_j, \theta_j)$ in order to really model the effect of passengers on the dwell times ?

- We show below that a convenient way is

$$\bar{w}(\rho) := \bar{w}_j(\rho) = \tilde{h}(\rho), \quad \forall j,$$
$$\theta(\rho) := \theta_j^k(\rho) = \tilde{w}^*(\rho)/\tilde{h}(\rho), \quad \forall j, k.$$

where $\tilde{h}$ and $\tilde{w}^*$ are respectively the asymptotic average time-headway and dwell time on platforms derived from the max-plus linear traffic model.

Control parameters $\bar{w}$ and θ

- At the stationary regime, server-stability condition is

$$\lambda \leq \alpha(\tilde{w}^*(\rho)/\tilde{h}(\rho))$$

- Then, for $\lambda_j^k = \tilde{\lambda}_j^k(\rho) := \alpha(\tilde{w}^*(\rho)/\tilde{h}(\rho)), \forall k, j$, we have

$$(\bar{w}_j, \theta_j^k) = (\tilde{h}(\rho), \lambda_j^k/\alpha_j^k) = (\tilde{h}(\rho), \tilde{w}^*(\rho)/\tilde{h}(\rho)).$$

- The dynamic system is max-plus linear, i.e. it behaves as if passengers do not have any effect on the train dynamics.

- Basing on that, we assume that $\tilde{\lambda}_j^k(\rho)$ is a threshold for λ_j^k, i.e. a lower bound for λ_j^k beyond which the passengers will have an effect on the train dynamics.
- We now fix, the parameters $(\bar{w}_j, \theta_j^k) = (\tilde{h}(\rho), \tilde{w}^*(\rho)/\tilde{h}(\rho))$ independent of λ_j^k.

1. If $\lambda_j^k = \tilde{\lambda}_j^k(\rho), \forall k, j$, we get the max-plus linear dynamic system, and the passengers do not have effect on the train dynamics.
2. If $\lambda_j^k \geq \tilde{\lambda}_j^k(\rho), \forall k, j$, we have

$$\theta_j^k = \tilde{w}^*(\rho)/\tilde{h}(\rho) = \tilde{\lambda}_j^k(\rho)/\alpha_j^k \leq \lambda_j^k/\alpha_j^k.$$

   - Then $\delta_j \leq 1$, and a dynamic programming system is obtained.
   - The dynamic system admits a stationary regime with a unique asymptotic average train time-headway $h(\rho)$ such that $h(\rho) > \tilde{h}$.
   - In this case, passengers have effect on the train dynamics, which can be measured by $h(\rho) - \tilde{h}(\rho)$.

3. If $\exists(k, j), \lambda_j^k < \tilde{\lambda}_j^k(\rho)$, we have

$$\theta_j^k = \tilde{w}^*(\rho)/\tilde{h}(\rho) = \tilde{\lambda}_j^k(\rho)/\alpha_j^k > \lambda_j^k/\alpha_j^k.$$

Then $\delta_j > 1$. We do not have a guarantee on the dynamic-stability of the dynamic system.

Control parameters $\bar{w}$ and θ

- In order to treat the dynamic-instability case (item (3) above) corresponding to $\lambda_j^k < \tilde{\lambda}_j^k(\rho)$, we take

$$\max(\lambda_j^k, \tilde{\lambda}_j^k(\rho)).$$

- The dwell time constraint is now

$$w_j^k \geq \begin{cases} \tilde{h}(\rho) - \frac{\alpha_j^k \tilde{w}^*(\rho)}{\max(\lambda_j^k, \tilde{\lambda}_j^k(\rho))\tilde{h}(\rho)} g_j^k & j \text{ is platform}, \\ 0 & \text{otherwise}. \end{cases}$$

- The dynamics are written

$$d_j^k = \max \begin{cases} d_{j-1}^{k-b_j} + r_j + \underline{w}_j, \\ (1 - \tilde{\delta}_j^k(\rho)) d_{j-1}^{k-b_j} + \tilde{\delta}_j^k(\rho) d_j^{k-1} + \dots \\ \quad + (1 - \tilde{\delta}_j^k(\rho)) r_j + \tilde{h}(\rho), \\ d_{j+1}^{k-\bar{b}_{j+1}} + \underline{s}_{j+1}, \end{cases}$$

where $\forall k, j, \rho$,

$$0 \leq \tilde{\delta}_j^k(\rho) = \frac{\alpha_j^k \tilde{w}^*(\rho)}{\max(\lambda_j^k, \tilde{\lambda}_j^k(\rho))\tilde{h}(\rho)} = \frac{\tilde{\lambda}_j^k(\rho)}{\max(\lambda_j^k, \tilde{\lambda}_j^k(\rho))} \leq 1.$$

Control parameters $\bar{w}$ and θ - Summary

We summarize the latter findings in the following result.
Theorem.
If $\tilde{\delta}_j^k(\rho)$ are independent of k for every ρ and j, then dynamic system

$$d_j^k = \max \begin{cases} d_{j-1}^{k-b_j} + r_j + \underline{w}_j, \\ (1-\tilde{\delta}_j^k(\rho))d_{j-1}^{k-b_j} + \tilde{\delta}_j^k(\rho)d_j^{k-1} + \dots \\ \quad +(1-\tilde{\delta}_j^k(\rho))r_j + \tilde{h}(\rho), \\ d_{j+1}^{k-\bar{b}_{j+1}} + \underline{s}_{j+1}, \end{cases}$$

where $\forall k, j, \rho$,

$$\begin{aligned} 0 \leq \tilde{\delta}_j^k(\rho) &= \frac{\alpha_j^k \tilde{w}^*(\rho)}{\max(\lambda_j^k, \tilde{\lambda}_j^k(\rho))\tilde{h}(\rho)} \\ &= \frac{\tilde{\lambda}_j^k(\rho)}{\max(\lambda_j^k, \tilde{\lambda}_j^k(\rho))} \leq 1. \end{aligned}$$

admits a stationary regime with a unique additive eigenvalue h, which coincides with the asymptotic average growth rate of the system, independent of the initial state d^0.
Moreover, We have

$$h \geq \tilde{h}.$$

Numerical example (1/3)

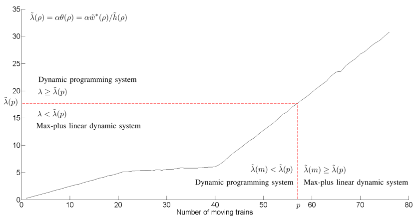


FIG.: Illustration of $\tilde\lambda(\rho)$, which is proportional here to the control parameter $\theta(\rho)$.

$$\tilde\lambda_j^k(\rho) = \alpha_j^k(\tilde w^*(\rho)/\tilde h(\rho)),$$

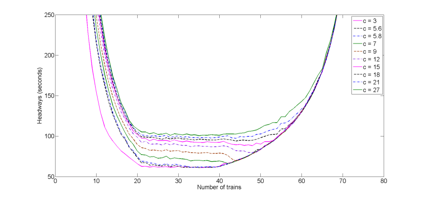


FIG.: Asymptotic average train time-headway in function of the number of moving trains. The average passenger arrivals on the platforms is equal to 1 times a factor c given in the figure.

Numerical example (2/3)

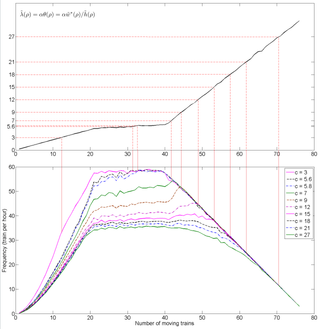


FIG.: Asymptotic average train frequency in function of the number of moving trains. The average arrival passenger on the platforms is equal to 1 times a factor *c* given in the figure.

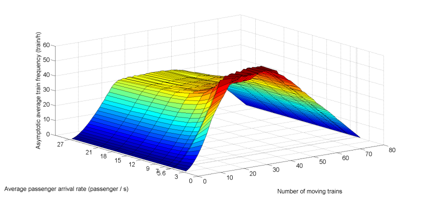


FIG.: Asymptotic average train frequency in function of the number of moving trains, and of the average arrival passenger on the platforms.

Numerical example (3/3)

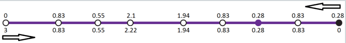


FIG.: Average arrival rates λ_j for every platform j, in passenger by second. The mean of those rates is 1.

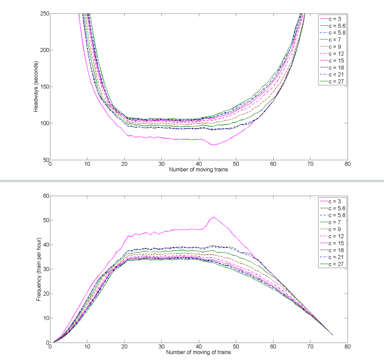


FIG.: Train time-headways and flows in function of the number of moving trains. The arrival passenger rates are varied by multiplication by factor c given in the figure.

Thank you for your attention

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